Acoustic propagation for realistic swirling isentropic flow in a slowly varying duct

James R. Mathews* **and N. Peake†**

University of Cambridge, Cambridge, CB3 0WA, UK

This paper considers the acoustic propagation for realistic swirling isentropic flow in a slowly varying duct. We extend previous studies by Rienstra and Cooper & Peake on the acoustic disturbances to allow for realistic swirling flow, and also apply the corrected Myers boundary condition which is derived for a curved duct. Results will be presented for realistic geometries and flows, and overcome any limitations from previous work.

I. Introduction

In this paper we consider the acoustic propagation for realistic isentropic swirling flow in a slowly varying duct. This has previously been considered by Rienstra[1] in the case of no swirl, and Cooper & Peake[2] and Cooper[3] for swirling flow, with the latter paper also including the effect of a base flow with varying entropy. However, all these papers consider simple flows which are not realistic and are chosen to simplify the mathematics. They also consider the acoustic potential, whereas we will consider the pressure and velocities. We will also use the corrected Myers boundary condition, which was derived in Masson et al[4] for a straight duct and we have extended to a duct with curvature. This showed that as soon as the base flow pressure gradient varies then the standard Myers boundary condition is incorrect.

We will follow a similar method to Cooper & Peake[2] and Cooper,[3] which in brief is as follows:

- Solve for the base flow using streamlines.
- Make the multiple scales approximation in a slowly varying duct.
- The leading order solution is equivalent to solving the usual disturbance problem in a straight duct.
- The slowly varying amplitude $N(X)$ is found by considering the disturbance problem at first order and using a secular condition.

A. Issues with realistic flow.

In the case of arbitrary realistic flow there are several difficulties with the method present in Cooper & Peake[2] and Cooper.[3] Firstly, when solving for the base flow, the method needs to be extended numerically for realistic flows. For example, the inversion of the streamfunction $\psi(r)$ will in general need to be done numerically. Secondly, in both the eigenvalue problem and the method for calculating the slowly varying amplitude $N(X)$ there is a $1/\Gamma$ term in Cooper & Peake[2] and Cooper,[3] where

$$\Gamma = \frac{1}{r}\frac{\partial}{\partial r}(rW_0), \tag{1}$$

with W_0 the leading order base flow swirl. For a flow of the form $W_0(r) = Ar + B/r$, $\Gamma = A$, but for realistic flows Γ may be zero, which clearly introduces singularities which are not physical. The governing equations need to be reformulated and the method rederived in order to avoid this singularity.

*Research Associate, Department of Engineering, jrm214@cam.ac.uk. Professional member.

†Professor, Department of Applied Mathematics and Theoretical Physics, np100@cam.ac.uk. Senior member.

II. Base flow

For solving the base flow, we use the same method as in Cooper & Peake[2] and Cooper,[3] but extend it numerically to allow arbitrary flow. We now allow the entropy to vary. Let the slowly varying duct be defined as

$$H(X) < r < D(X), \tag{2}$$

where $X = \varepsilon x$ for some small parameter ε, assumed to be the gradient of the duct. We assume the base flow is of the form

$$U(X, r; \varepsilon) = U_0(X, r) + \mathcal{O}(\varepsilon^2), \tag{3a}$$
$$V(X, r; \varepsilon) = \varepsilon V_1(X, r) + \mathcal{O}(\varepsilon^3), \tag{3b}$$
$$W(X, r; \varepsilon) = W_0(X, r) + \mathcal{O}(\varepsilon^2), \tag{3c}$$
$$\rho(X, r; \varepsilon) = \rho_0(X, r) + \mathcal{O}(\varepsilon^2), \tag{3d}$$
$$P(X, r; \varepsilon) = P_0(X, r) + \mathcal{O}(\varepsilon^2), \tag{3e}$$
$$C(X, r; \varepsilon) = c_0(X, r) + \mathcal{O}(\varepsilon^2), \tag{3f}$$
$$S(X, r; \varepsilon) = S_0(X, r) + \mathcal{O}(\varepsilon^2). \tag{3g}$$

We introduce the streamline $\psi(X, r)$ and solve the following coupled problem between the streamline $\psi(X, r)$ and density $\rho_0(X, r)$:

$$\frac{\partial}{\partial r}\left(\frac{1}{r\rho_0}\frac{\partial \psi}{\partial r}\right) = r\rho_0 \mathcal{H}'(\psi) - \rho_0 \mathcal{C}'(\psi)\frac{\mathcal{C}(\psi)}{r} - \frac{rP_0}{c_p - c_v}\mathscr{S}'(\psi), \tag{4a}$$
$$\frac{1}{2}\frac{1}{r^2\rho_0^2}\left(\frac{\partial \psi}{\partial r}\right)^2 + \frac{\mathcal{C}^2(\psi)}{2r^2} + \frac{\rho_0^{\gamma-1}}{\gamma - 1}e^{\mathscr{S}/c_v} = \mathcal{H}(\psi), \tag{4b}$$

with the boundary conditions

$$\psi(X, H(X)) = 0, \quad \psi(X, D(X)) = \psi(0, D(0)), \tag{5}$$

where $\mathcal{H}$, $\mathcal{C}$ and $\mathscr{S}$ are determined from the inlet conditions at $X = 0$. The only X dependence is through the boundary condition.

To solve this system of equations, we use Chebfun.[5] We solve the systems of equations separately for each X value. We use an initial guess for $\rho_0(X, r)$ (from $\rho_0(X_{\text{prev}}, r)$), and then solve Eq. (4a) with the two boundary conditions in Eq. (5). This then gives us ψ. We then use this $\psi(X, r)$ and solve the algebraic equation in Eq. (4b) for the density $\rho_0(X, r)$. We iterate the procedure until we get convergence.

Once we have calculated $\psi(X, r)$ and $\rho_0(X, r)$, we can calculate the other quantities:

$$U_0(X, r) = \frac{1}{r\rho_0(X, r)}\frac{\partial \psi}{\partial r}(X, r), \quad V_1(X, r) = -\frac{1}{r\rho_0(X, r)}\frac{\partial \psi}{\partial X}(X, r), \tag{6}$$
$$W_0(X, r) = \frac{C(\psi(X, r))}{r}, \quad S_0 = \mathscr{S}(\psi(X, r)), \quad c_0^2(X, r) = \exp\left(\frac{\mathscr{S}(\psi(X, r))}{c_v}\right)\rho_0(X, r)^{\gamma-1}.$$

For the case of homentropic flow we have that $\mathscr{S}$ and S_0 is identically zero, and the expression for the speed of sound simplifies.

A. Solving for the functions $\mathcal{H}$, $\mathcal{C}$ and $\mathscr{S}$ at the inlet

We assume the values of $U_0(r)$, $W_0(r)$ and $V_1 = 0$ are known at the inlet $X = 0$, and additionally we have to prescribe the density or entropy at $X = 0$.

1. *Entropy known*

Assume that the entropy $S_0(r)$ is specified at the inlet $X = 0$. From conservation of momentum we have

$$\frac{1}{\rho_0}\frac{\mathrm{d}P_0}{\mathrm{d}r}(r) = \frac{W_0^2(r)}{r}, \tag{7}$$

while the constitutive relation between pressure and density gives

$$P_0 = \frac{\rho_0^\gamma}{\gamma} e^{S_0/c_v}. \tag{8}$$

Combining these gives a differential equation for density:

$$\frac{1}{\gamma - 1}\frac{\mathrm{d}(\rho_0^{\gamma-1})}{\mathrm{d}r}(r) + \frac{\rho_0^{\gamma-1}(r)}{c_p}S_0'(r) = \frac{W_0^2(r)}{r}e^{-S_0(r)/c_v}. \tag{9}$$

Solving this using an integrating factor then gives

$$\rho_0(r) = e^{-S_0(r)/c_p}\left((\gamma-1)\int_{H(0)}^{r}\frac{W_0^2(\tau)}{\tau}e^{-S_0(\tau)/c_p}\mathrm{d}\tau + K\right)^{1/(\gamma-1)}. \tag{10}$$

We choose arbitrarily the constant of integration K such that $\rho_0(H(0)) = 1$.

2. *Density known*

If the density $\rho_0(r)$ is prescribed at the inlet instead of $S_0(r)$, then we can determine $S_0(r)$ easily. From Eq. (7) we can calculate that

$$P_0(r) = \frac{1}{\gamma} + \int_{H(0)}^{r}\rho_0(\tau)\frac{W_0^2(\tau)}{\tau}\mathrm{d}\tau, \tag{11}$$

although the choice of constant is arbitrary. Since both the base flow pressure and density are now known at the inlet, we can use Eq. (8) to calculate the entropy:

$$S_0(r) = c_v \log\left(\frac{\gamma P_0(r)}{\rho_0^{\gamma-1}(r)}\right). \tag{12}$$

3. *Final result*

Once we have $\rho_0(r)$ at $X = 0$, we can determine $\psi(r)$ at $X = 0$ from

$$\psi(r) = \int_{H(0)}^{r} s\rho_0(s)U_0(s)\mathrm{d}s, \tag{13}$$

so that the streamline is zero at the inner duct wall. We then invert this equation to get $r(\psi)$.

We thus find that

$$\mathcal{C}(\psi) = r(\psi)W_0(r(\psi)), \tag{14}$$

$$\mathcal{H}(\psi) = \frac{U_0^2(r(\psi)) + W_0^2(r(\psi))}{2} + \frac{\rho_0^{\gamma-1}(r(\psi))}{\gamma-1}e^{S_0(r(\psi))/c_v}, \tag{15}$$

$$\mathscr{S}(\psi) = S_0(r(\psi)). \tag{16}$$

B. Results

For initial testing purposes we consider the flow at the inlet to be given by

$$U_0(0,r) = 0.25 + 0.25r - 0.3r^3, \quad V_0(0,r) = 0, \quad W_0(0,r) = 0.2 + 0.1r + 0.2/r \tag{17}$$

More realistic flows such as those in Mathews & Peake[6] and Masson et al[4] are considered in Section B. We validating our results by comparing results for the flow $U_0(0,r) = 0.3$, $W_0(0,r) = 0.3r$ to those in Cooper,[3] with excellent agreement.

We consider a slowly varying, contracting duct based on Cooper,[2] which is given by:

$$H(X) = 0.5482 + 0.05\tanh(2X-2),\ D(X) = 0.9518 - 0.05\tanh(2X-2), \tag{18}$$

for $0 < X < 2$. The base flow is given in Figure 1 when we assume the inlet density at $X = 0$ to be $\rho_0(0,r) = 1$.

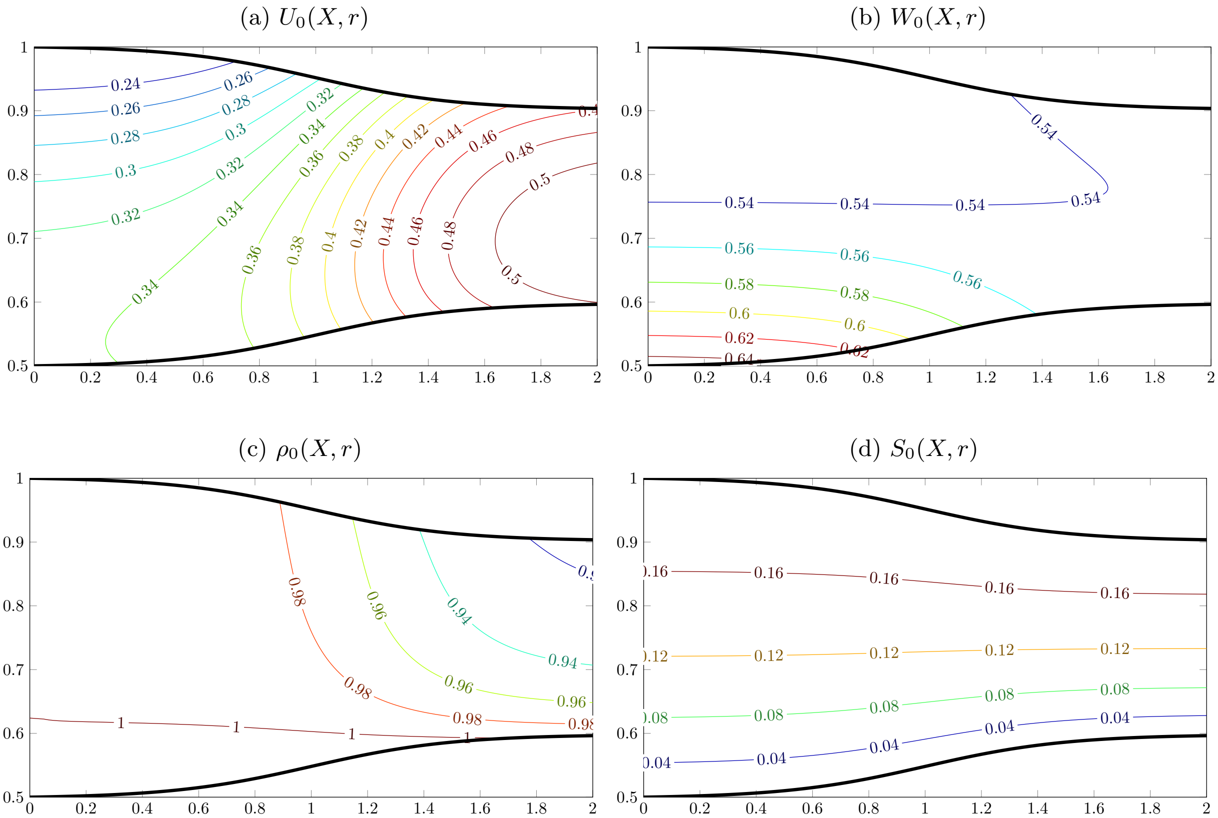

Figure 1: Plot of the base flow, with inlet flow given in Eq. (17) and $\rho_0(0,r) = 1$, and the duct defined in Eq. (18).

III. Corrected Myers boundary condition

Based on the result from Masson et al,[4] we expect the corrected Myers boundary condition to be

$$-(\boldsymbol{v}\cdot\boldsymbol{n_j}) = (-\mathrm{i}\omega + \boldsymbol{V}\cdot\boldsymbol{\nabla} - \boldsymbol{n_j}\cdot(\boldsymbol{n_j}\cdot\boldsymbol{\nabla}\boldsymbol{V}))\left(\frac{p}{\mathrm{i}\omega Z_j + \boldsymbol{n_j}\cdot\nabla P}\right), \tag{19}$$

where $\boldsymbol{n_j}$ is the outward unit normal to the duct, and $\boldsymbol{V}$ and P are the base flow velocity and pressure respectively. Note the extra term $\boldsymbol{n_j}\cdot\nabla P$ compared to the standard Myers' boundary condition in a curved duct.

Although this is hypothesised boundary condition, we will see later that to the order we require it, it is identical to the boundary condition given in Masson et al,[4] with the $\boldsymbol{n_j}\cdot\nabla P$ replaced by a $\frac{\partial P}{\partial r}$ term.

IV. Multiple scales expansion

We introduce the multiple scales expansion for the pressure, velocity and entropy perturbations:

$$(p,u,v,w,s)(x,r,\theta,t;\varepsilon) = (\mathcal{P},\mathcal{U},\mathcal{V},\mathcal{W},\mathcal{S})(X,r;\varepsilon)\exp\left(\frac{\mathrm{i}}{\varepsilon}\int^{X} k(\eta)\mathrm{d}\eta + \mathrm{i}m\theta - \mathrm{i}\omega t\right), \tag{20}$$

and then expand the slowly varying amplitudes as functions of ε:

$$(\mathcal{P},\mathcal{U},\mathcal{V},\mathcal{W},\mathcal{S})(X,r;\varepsilon) = (\mathcal{P}_0,\mathcal{U}_0,\mathcal{V}_0,\mathcal{W}_0,\mathcal{S}_0)(X,r) + \varepsilon(\mathcal{P}_1,\mathcal{U}_1,\mathcal{V}_1,\mathcal{W}_1,\mathcal{S}_1)(X,r) + \mathcal{O}(\varepsilon^2). \tag{21}$$

We have the same sign convection as in Cooper,[3] but the opposite sign convection to Cooper & Peake.[2]

A. Leading order solution

Letting $\boldsymbol{\psi}_0 = (\mathcal{P}_0, \mathrm{i}\mathcal{U}_0, \mathcal{V}_0, \mathrm{i}\mathcal{W}_0, \mathcal{S}_0)$, the eigenvalue problem at leading order is given by

$$\mathcal{L}\boldsymbol{\psi}_0 = (\mathscr{L} - k(X)\mathscr{M})\boldsymbol{\psi}_0 = \mathbf{0}, \tag{22}$$

with boundary conditions given by

$$\mathcal{V}_0(H(X)) = \frac{\Lambda(H(X))}{\omega Z_H^\ddagger}\mathcal{P}_0(H(X)), \quad \mathcal{V}_0(D(X)) = -\frac{\Lambda(D(X))}{\omega Z_D^\ddagger}\mathcal{P}_0(D(X)), \tag{23}$$

where

$$\Lambda(r) = k(X)U_0(X,r) + m\frac{W_0(X,r)}{r} - \omega, \tag{24}$$

and $Z_j^\ddagger$ are the corrected impedances from Masson et al:[4]

$$Z_H^\ddagger = Z_H(X) + \frac{\mathrm{i}}{\omega}\frac{\partial P_0}{\partial r}(X, H(X)), \tag{25}$$

$$Z_D^\ddagger = Z_D(X) - \frac{\mathrm{i}}{\omega}\frac{\partial P_0}{\partial r}(X, D(X)). \tag{26}$$

In the case of a hard-walled duct, we have $Z_j^\ddagger = Z_j = \infty$, and hence the boundary conditions are given by $\mathcal{V}_0 = 0$ at the duct walls. The matrices are given by

$$\mathscr{L} = \begin{bmatrix} -\frac{\mathrm{i}\omega_m}{c_0^2} & 0 & \rho_0\frac{\partial}{\partial r} + \frac{\rho_0}{r} + \frac{\rho_0 W_0^2}{rc_0^2} & \frac{m\rho_0}{r} & 0 \\ 0 & -\rho_0\omega_m & \rho_0\frac{\partial U_0}{\partial r} & 0 & 0 \\ -\frac{W_0^2}{rc_0^2} + \frac{\partial}{\partial r} & 0 & -\mathrm{i}\rho_0\omega_m & \frac{2\mathrm{i}\rho_0 W_0}{r} & \frac{\rho_0 W_0^2}{rc_p} \\ \frac{\mathrm{i}m}{r} & 0 & \rho_0\Gamma & -\rho_0\omega_m & 0 \\ 0 & 0 & \frac{\partial S_0}{\partial r} & 0 & -\omega_m \end{bmatrix}, \tag{27}$$

and

$$\mathscr{M} = \begin{bmatrix} -\frac{\mathrm{i}U_0}{c_0^2} & -\rho_0 & 0 & 0 & 0 \\ -\mathrm{i} & -\rho_0 U_0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i}\rho_0 U_0 & 0 & 0 \\ 0 & 0 & 0 & -\rho_0 U_0 & 0 \\ 0 & 0 & 0 & 0 & -U_0 \end{bmatrix}, \tag{28}$$

where $\omega_m = \omega - mW_0/r$ and $\Gamma = (1/r)(\partial/\partial r)(rW_0)$. The derivation is given in Section B. Although it is possible to rearrange the equations in such a way that $\mathscr{M}$ is the identity matrix (see for example Posson & Peake[7]), it is not necessary to solve the problem and makes $\mathscr{L}$ more complicated. In the case of homentropic flow then we can simply remove the last row and column of $\mathscr{L}$ and $\mathscr{M}$, to give a four by four system.

The solution to the problem is found numerically, and again we use Chebfun to calculate $k(X)$ and $\boldsymbol{\psi}_0$ at each value of X. Since the eigenfunction is only determined up to a constant (which could vary with X), we have that

$$\boldsymbol{\psi}_0 = N(X)\widehat{\boldsymbol{\psi}_0}, \tag{29}$$

where $\widehat{\boldsymbol{\psi}_0}$ has been normalised. It remains to calculate $N(X)$ to determine the complete leading order solution.

Note that we have considered the linearised Euler equation, rather than rewriting the problem in terms of a potential ϕ as in Cooper & Peake.[2] As a result our leading order solution only agrees with the leading order solution from Cooper & Peake[2] to $\mathcal{O}(\varepsilon)$. If we use the method in Cooper[3] but don't divide by Γ then we would instead have a six by six system.

B. First order solution

The first order solution is harder to calculate. We begin with the governing equations with the base flow having dependence on x and r, and being in all three directions. Let the density perturbation be λ (with the base flow density ρ), then we note that the density and pressure perturbations are linked by

$$\lambda = \frac{p}{C^2} - \frac{\rho}{c_p} s. \tag{30}$$

Using the mass equation and this relationship (or equivalently the energy equation

$$\frac{\partial \underline{p}}{\partial t} + \underline{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \underline{p} + \gamma \underline{p} \left(\boldsymbol{\nabla} \cdot \underline{\boldsymbol{u}} \right) = 0, \tag{31}$$

where the underlined variables are the base flow plus perturbation), we get

$$\frac{1}{C^2} \frac{\widetilde{D_0 p}}{Dt} + \frac{1}{C^2} \frac{\partial P}{\partial r} v + \rho \left(\frac{\partial u}{\partial x} + \frac{v}{r} + \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) + u \frac{1}{C^2} \frac{\partial P}{\partial x} + \frac{\gamma p}{C^2} \frac{\partial U_x}{\partial x} + \frac{\gamma p}{r C^2} \frac{\partial}{\partial r} \left(r U_r \right) = 0. \tag{32a}$$

The momentum equation and entropy equation then give

$$\rho \frac{\widetilde{D_0 u}}{Dt} + \rho v \frac{\partial U_x}{\partial r} + \frac{\partial p}{\partial x} + \left(\frac{p}{C^2} - \frac{\rho s}{c_p} \right) \left(U_x \frac{\partial U_x}{\partial x} + U_r \frac{\partial U_x}{\partial r} \right) + \rho u \frac{\partial U_x}{\partial x} = 0, \tag{32b}$$

$$\rho \frac{\widetilde{D_0 v}}{Dt} - \frac{2 \rho U_\theta}{r} w - \frac{U_\theta^2}{r C^2} p + \frac{\rho U_\theta^2}{r c_p} s + \frac{\partial p}{\partial r} + \left(\frac{p}{C^2} - \frac{\rho s}{c_p} \right) \left(U_x \frac{\partial U_r}{\partial x} + U_r \frac{\partial U_r}{\partial r} \right) + \rho u \frac{\partial U_r}{\partial x} + \rho v \frac{\partial U_r}{\partial r} = 0, \tag{32c}$$

$$\rho \frac{\widetilde{D_0 w}}{Dt} + \rho \frac{1}{r} \frac{\partial}{\partial r} \left(r U_\theta \right) v + \frac{1}{r} \frac{\partial p}{\partial \theta} + \left(\frac{p}{C^2} - \frac{\rho s}{c_p} \right) \left(U_x \frac{\partial U_\theta}{\partial x} + U_r \frac{\partial U_\theta}{\partial r} + \frac{U_r U_\theta}{r} \right) + \rho u \frac{\partial U_\theta}{\partial x} + \frac{\rho U_r}{r} w = 0, \tag{32d}$$

$$\frac{\widetilde{D_0 s}}{Dt} + v \frac{\partial S}{\partial r} + u \frac{\partial S}{\partial x} = 0, \tag{32e}$$

where

$$\frac{\widetilde{D_0}}{Dt} = \frac{\partial}{\partial t} + U_x \frac{\partial}{\partial x} + U_r \frac{\partial}{\partial r} + \frac{U_\theta}{r} \frac{\partial}{\partial \theta}. \tag{33}$$

The extra terms compared to assuming no x dependence and no radial velocity in the base flow have been highlighted in blue. Now expanding the base flow as in Eq. (3), and relating the x derivative to the X derivative by a factor of ε then gives

$$\frac{1}{c_0^2} \frac{D_0 p}{Dt} + \frac{1}{c_0^2} \frac{\partial P_0}{\partial r} v + \rho_0 \left(\frac{\partial u}{\partial x} + \frac{v}{r} + \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) = - \varepsilon \left[u \frac{1}{c_0^2} \frac{\partial P_0}{\partial X} + \frac{\gamma p}{c_0^2} \frac{\partial U_0}{\partial X} + \frac{\gamma p}{r c_0^2} \frac{\partial}{\partial r} \left(r V_1 \right) + \frac{V_1}{c_0^2} \frac{\partial p}{\partial r} \right] + \mathcal{O}(\varepsilon^2), \tag{34a}$$

$$\rho_0 \frac{D_0 u}{Dt} + \rho_0 v \frac{\partial U_0}{\partial r} + \frac{\partial p}{\partial x} = - \varepsilon \left[\left(\frac{p}{C^2} - \frac{\rho_0 s}{c_p} \right) \left(U_0 \frac{\partial U_0}{\partial X} + V_1 \frac{\partial U_0}{\partial r} \right) + \rho_0 u \frac{\partial U_0}{\partial X} + \rho_0 V_1 \frac{\partial u}{\partial r} \right] + \mathcal{O}(\varepsilon^2), \tag{34b}$$

$$\rho_0 \frac{D_0 v}{Dt} - \frac{2 \rho_0 W_0}{r} w - \frac{W_0^2}{r c_0^2} p + \frac{\rho_0 W_0^2}{r c_p} s + \frac{\partial p}{\partial r} = - \varepsilon \left[\rho_0 v \frac{\partial V_1}{\partial r} + \rho_0 V_1 \frac{\partial v}{\partial r} \right] + \mathcal{O}(\varepsilon^2), \tag{34c}$$

$$\rho_0 \frac{D_0 w}{Dt} + \rho_0 \Gamma v + \frac{1}{r} \frac{\partial p}{\partial \theta} = - \varepsilon \left[\left(\frac{p}{C^2} - \frac{\rho_0 s}{c_p} \right) \left(U_0 \frac{\partial W_0}{\partial X} + V_1 \frac{\partial W_0}{\partial r} + \frac{V_1 W_0}{r} \right) + \rho_0 u \frac{\partial W_0}{\partial X} + \frac{\rho_0 V_1}{r} w + \rho_0 V_1 \frac{\partial w}{\partial r} \right] + \mathcal{O}(\varepsilon^2), \tag{34d}$$

$$\frac{D_0 s}{Dt} + v \frac{\partial S_0}{\partial r} = - \varepsilon \left[u \frac{\partial S_0}{\partial X} + V_1 \frac{\partial s}{\partial r} \right] + \mathcal{O}(\varepsilon^2), \tag{34e}$$

where

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x} + \frac{W_0}{r} \frac{\partial}{\partial \theta}. \tag{35}$$

We then use the expansion in Eq. (20) for the perturbations, and note that if $\phi = \{p, u, v, w, s\}$ and $\Phi = \{\mathcal{P}, \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{S}\}$ then

$$\frac{\partial \phi}{\partial t} = -\mathrm{i}\omega\left(\Phi_0 + \varepsilon\Phi_1\right), \quad \frac{\partial \phi}{\partial r} = \frac{\partial \Phi_0}{\partial r} + \varepsilon\frac{\partial \Phi_1}{\partial r},$$
$$\frac{\partial \phi}{\partial \theta} = \mathrm{i}m\left(\Phi_0 + \varepsilon\Phi_1\right), \quad \frac{\partial \phi}{\partial x} = \varepsilon\frac{\partial \Phi_0}{\partial X} + \mathrm{i}k(X)\left(\Phi_0 + \varepsilon\Phi_1\right), \tag{36}$$

to $\mathcal{O}(\varepsilon^2)$. Hence, Eq. (34) now reads

$$\frac{\mathrm{i}\Lambda}{c_0^2}\left(\mathcal{P}_0 + \varepsilon\mathcal{P}_1\right) + \frac{U_0}{c_0^2}\varepsilon\frac{\partial \mathcal{P}_0}{\partial X} + \frac{1}{c_0^2}\frac{\partial P_0}{\partial r}\left(\mathcal{V}_0 + \varepsilon\mathcal{V}_1\right) + \rho_0\Big[\mathrm{i}k\left(\mathcal{U}_0 + \varepsilon\mathcal{U}_1\right) + \varepsilon\frac{\partial \mathcal{U}_0}{\partial X} + \frac{1}{r}\left(\mathcal{V}_0 + \varepsilon\mathcal{V}_1\right) + \frac{\partial \mathcal{V}_0}{\partial r} \tag{37a}$$
$$+ \varepsilon\frac{\partial \mathcal{V}_1}{\partial r} + \frac{\mathrm{i}m}{r}\left(\mathcal{W}_0 + \varepsilon\mathcal{W}_1\right)\Big] - \varepsilon\left[\mathcal{U}_0\frac{1}{c_0^2}\frac{\partial P_0}{\partial X} + \frac{\gamma\mathcal{P}_0}{c_0^2}\frac{\partial U_0}{\partial X} + \frac{\gamma\mathcal{P}_0}{rc_0^2}\frac{\partial}{\partial r}\left(rV_1\right) + \frac{V_1}{c_0^2}\frac{\partial \mathcal{P}_0}{\partial r}\right] + \mathcal{O}(\varepsilon^2),$$

$$\rho_0\mathrm{i}\Lambda\left(\mathcal{U}_0 + \varepsilon\mathcal{U}_1\right) + \rho_0\varepsilon U_0\frac{\partial \mathcal{U}_0}{\partial X} + \rho_0\left(\mathcal{V}_0 + \varepsilon\mathcal{V}_1\right)\frac{\partial U_0}{\partial r} + \mathrm{i}k\left(\mathcal{P}_0 + \varepsilon\mathcal{P}_1\right) + \varepsilon\frac{\partial \mathcal{P}_0}{\partial X} \tag{37b}$$
$$= -\varepsilon\left[\left(\frac{\mathcal{P}_0}{c_0^2} - \frac{\rho_0\mathcal{S}_0}{c_p}\right)\left(U_0\frac{\partial U_0}{\partial X} + V_1\frac{\partial U_0}{\partial r}\right) + \rho_0\mathcal{U}_0\frac{\partial U_0}{\partial X} + \rho_0 V_1\frac{\partial \mathcal{U}_0}{\partial r}\right] + \mathcal{O}(\varepsilon^2),$$

$$\rho_0\mathrm{i}\Lambda\left(\mathcal{V}_0 + \varepsilon\mathcal{V}_1\right) + \rho_0\varepsilon U_0\frac{\partial \mathcal{V}_0}{\partial X} - \frac{2\rho_0 W_0}{r}\left(\mathcal{W}_0 + \varepsilon\mathcal{W}_1\right) - \frac{W_0^2}{rc_0^2}\left(\mathcal{P}_0 + \varepsilon\mathcal{P}_1\right) \tag{37c}$$
$$+ \frac{\rho_0 W_0^2}{rc_p}\left(\mathcal{S}_0 + \varepsilon\mathcal{S}_1\right) + \frac{\partial \mathcal{P}_0}{\partial r} + \varepsilon\frac{\partial \mathcal{P}_1}{\partial r} = -\varepsilon\left[\rho_0\mathcal{V}_0\frac{\partial V_1}{\partial r} + \rho_0 V_1\frac{\partial \mathcal{V}_0}{\partial r}\right] + \mathcal{O}(\varepsilon^2),$$

$$\rho_0\mathrm{i}\Lambda\left(\mathcal{W}_0 + \varepsilon\mathcal{W}_1\right) + \rho_0\varepsilon U_0\frac{\partial \mathcal{W}_0}{\partial X} + \rho_0\Gamma\left(\mathcal{V}_0 + \varepsilon\mathcal{V}_1\right) + \frac{\mathrm{i}m}{r}\left(\mathcal{P}_0 + \varepsilon\mathcal{P}_1\right) \tag{37d}$$
$$= -\varepsilon\left[\left(\frac{\mathcal{P}_0}{c_0^2} - \frac{\rho_0\mathcal{S}_0}{c_p}\right)\left(U_0\frac{\partial W_0}{\partial X} + V_1\frac{\partial W_0}{\partial r} + \frac{V_1 W_0}{r}\right) + \rho_0\mathcal{U}_0\frac{\partial W_0}{\partial X} + \frac{\rho_0 V_1}{r}\mathcal{W}_0 + \rho_0 V_1\frac{\partial \mathcal{W}_0}{\partial r}\right] + \mathcal{O}(\varepsilon^2),$$

$$\mathrm{i}\Lambda(\mathcal{S}_0 + \varepsilon\mathcal{S}_1) + \varepsilon U_0\frac{\partial \mathcal{S}_0}{\partial X} + \frac{\partial S_0}{\partial r}(\mathcal{V}_0 + \varepsilon\mathcal{V}_1) = -\varepsilon\left[\mathcal{U}_0\frac{\partial S_0}{\partial X} + V_1\frac{\partial \mathcal{S}_0}{\partial r}\right] + \mathcal{O}(\varepsilon^2). \tag{37e}$$

Hence to leading order we get $\mathcal{L}\boldsymbol{\psi}_0 = 0$, where we have additionally used

$$\frac{\partial P_0}{\partial r} = \frac{\rho_0 W^2}{r} + \rho_0\left(V\frac{\partial V}{\partial r} + U\frac{\partial V}{\partial x}\right) = \frac{\rho_0 W_0^2}{r} + \mathcal{O}(\varepsilon^2), \tag{38}$$

which follows since the base flow satisfies the radial momentum equation. Letting $\boldsymbol{\psi}_1 = (\mathcal{P}_1, \mathrm{i}\mathcal{U}_1, \mathcal{V}_1, \mathrm{i}\mathcal{W}_1, \mathcal{S}_1)$, the disturbance problem at first order reads

$$\mathcal{L}\boldsymbol{\psi}_1 = \boldsymbol{f}, \tag{39}$$

where $\boldsymbol{f} = (f_{\mathcal{P}}, f_{\mathcal{U}}, f_{\mathcal{V}}, f_{\mathcal{W}}, f_{\mathcal{S}})$ is given by

$$f_{\mathcal{P}} = -\frac{U_0}{c_0^2}\frac{\partial \mathcal{P}_0}{\partial X} - \rho_0\frac{\partial \mathcal{U}_0}{\partial X} - \mathcal{U}_0\frac{1}{c_0^2}\frac{\partial P_0}{\partial X} - \frac{\gamma\mathcal{P}_0}{c_0^2}\frac{\partial U_0}{\partial X} - \frac{\gamma\mathcal{P}_0}{rc_0^2}\frac{\partial}{\partial r}\left(rV_1\right) - \frac{V_1}{c_0^2}\frac{\partial \mathcal{P}_0}{\partial r}, \tag{40a}$$

$$f_{\mathcal{U}} = -\rho_0 U_0\frac{\partial \mathcal{U}_0}{\partial X} - \frac{\partial \mathcal{P}_0}{\partial X} - \left(\frac{\mathcal{P}_0}{c_0^2} - \frac{\rho_0\mathcal{S}_0}{c_p}\right)\left(U_0\frac{\partial U_0}{\partial X} + V_1\frac{\partial U_0}{\partial r}\right) - \rho_0\mathcal{U}_0\frac{\partial U_0}{\partial X} - \rho_0 V_1\frac{\partial \mathcal{U}_0}{\partial r}, \tag{40b}$$

$$f_{\mathcal{V}} = -\rho_0 U_0\frac{\partial \mathcal{V}_0}{\partial X} - \rho_0\frac{\partial}{\partial r}\left(V_1\mathcal{V}_0\right), \tag{40c}$$

$$f_{\mathcal{W}} = -\rho_0 U_0\frac{\partial \mathcal{W}_0}{\partial X} - \left(\frac{\mathcal{P}_0}{c_0^2} - \frac{\rho_0\mathcal{S}_0}{c_p}\right)\left(U_0\frac{\partial W_0}{\partial X} + V_1\frac{\partial W_0}{\partial r} + \frac{V_1 W_0}{r}\right) - \rho_0\mathcal{U}_0\frac{\partial W_0}{\partial X} - \frac{\rho_0 V_1}{r}\mathcal{W}_0 - \rho_0 V_1\frac{\partial \mathcal{W}_0}{\partial r}, \tag{40d}$$

$$f_{\mathcal{S}} = -U_0\frac{\partial \mathcal{S}_0}{\partial X} - \mathcal{U}_0\frac{\partial S_0}{\partial X} - V_1\frac{\partial \mathcal{S}_0}{\partial r}. \tag{40e}$$

1. *First order boundary condition*

We use the boundary condition in Eq. (19), and for the slowly varying duct we have that

$$\boldsymbol{n_j} = \pm \frac{-\boldsymbol{e_r} + \varepsilon \frac{\partial R_j}{\partial X} \boldsymbol{e_x}}{\left(1+\varepsilon^2 \left(\frac{\partial R_j}{\partial X}\right)^2\right)^{1/2}} = \pm \left(-\boldsymbol{e_r} + \varepsilon \frac{\partial R_j}{\partial X} \boldsymbol{e_x}\right) + \mathcal{O}(\varepsilon^2), \tag{41}$$

with $\pm$ corresponding to the inner $(+)$ and outer $(-)$ walls. We can calculate that

$$\boldsymbol{n_j} \cdot (\boldsymbol{n_j} \cdot \boldsymbol{\nabla V}) = \varepsilon \left(\frac{\partial V_1}{\partial r} - \frac{\partial R_j}{\partial X}\frac{\partial U_0}{\partial r}\right) + \mathcal{O}(\varepsilon^2), \tag{42}$$

and also that

$$\boldsymbol{n}_j \cdot \nabla P = \mp \frac{\partial P_0}{\partial r} + \mathcal{O}(\varepsilon^2), \tag{43}$$

and thus

$$\frac{1}{\mathrm{i}\omega Z_j + \boldsymbol{n}_j \cdot \nabla P} = \frac{1}{\mathrm{i}\omega Z_j^{\ddagger}} + \mathcal{O}(\varepsilon^2). \tag{44}$$

Hence, the boundary condition becomes

$$\pm \left[\mathcal{V} - \varepsilon \frac{\partial R_j}{\partial X}\mathcal{U}\right] = \mathrm{i}\Lambda \left(\frac{\mathcal{P}}{\mathrm{i}\omega Z_j^{\ddagger}}\right) + \varepsilon \left[U_0 \frac{\partial}{\partial X} + V_1 \frac{\partial}{\partial r} - \frac{\partial V_1}{\partial r} + \frac{\partial R_j}{\partial X}\frac{\partial U_0}{\partial r}\right] \left(\frac{\mathcal{P}}{\mathrm{i}\omega Z_j^{\ddagger}}\right). \tag{45}$$

At leading order we therefore recover the boundary condition in Eq. (23). At first order the boundary condition reads

$$\pm \mathcal{V}_1 - \mathrm{i}\Lambda \left(\frac{\mathcal{P}_1}{\mathrm{i}\omega Z_j^{\ddagger}}\right) = \pm \frac{\partial R_j}{\partial X}\mathcal{U}_0 + \left[U_0 \frac{\partial}{\partial X} + V_1 \frac{\partial}{\partial r} - \frac{\partial V_1}{\partial r} + \frac{\partial R_j}{\partial X}\frac{\partial U_0}{\partial r}\right] \left(\frac{\mathcal{P}_0}{\mathrm{i}\omega Z_j^{\ddagger}}\right). \tag{46}$$

V. Slowly varying function $N(X)$

To determine the function $N(X)$, we will use a secular condition from considering the problem at first order.

A. Adjoint matrix

Given $\mathcal{L}$ and the boundary conditions Eq. (23) we wish to find the adjoint operator $\mathcal{L}^{\dagger}$ and associated boundary conditions. The adjoint operator satisfies

$$\langle \boldsymbol{\mathcal{Y}}, \mathcal{L}\boldsymbol{\mathcal{X}} \rangle = \langle \mathcal{L}^{\dagger}\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{X}} \rangle, \tag{47}$$

with $\boldsymbol{\mathcal{X}}$ satisfying the boundary conditions Eq. (23) and $\boldsymbol{\mathcal{Y}}$ satisfying the adjoint boundary conditions. The inner product between $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{X}}$ will be

$$\langle \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{X}} \rangle = \int_{H(X)}^{D(X)} \sum_{n=1}^{5} \mathcal{Y}_n^{\star} \mathcal{X}_n r \mathrm{d}r, \tag{48}$$

with * the complex conjugate. Letting $\mathcal{X} = \boldsymbol{\psi}_0$, then we can calculate that

$$0 = \langle \boldsymbol{\mathcal{Y}}, \mathcal{L}\boldsymbol{\psi}_0 \rangle = \langle \mathcal{L}^{\dagger}\boldsymbol{\mathcal{Y}}, \boldsymbol{\psi}_0 \rangle + \Big[r\rho_0 \mathcal{Y}_1^{\star}\mathcal{V}_0 + r\mathcal{Y}_3^{\star}\mathcal{P}_0\Big]_{H(X)}^{D(X)}, \tag{49}$$

where

$$\mathcal{L}^\dagger = \begin{bmatrix} \frac{\mathrm{i}(\omega_m - k^\star U_0)}{c_0^2} & -\mathrm{i}k^\star & -\frac{W_0^2}{rc_0^2} - \left[\frac{1}{r} + \frac{\partial}{\partial r}\right] & -\frac{\mathrm{i}m}{r} & 0 \\ k^\star \rho_0 & \rho_0(k^\star U_0 - \omega_m) & 0 & 0 & 0 \\ \frac{\rho_0 W_0^2}{rc_0^2} - \frac{\partial \rho_0}{\partial r} - \rho_0 \frac{\partial}{\partial r} & \rho_0 \frac{\partial U_0}{\partial r} & \mathrm{i}\rho_0(\omega_m - k^\star U_0) & \rho_0 \Gamma & \frac{\partial S_0}{\partial r} \\ \frac{m\rho_0}{r} & 0 & -\frac{2\mathrm{i}\rho_0 W_0}{r} & \rho_0(k^\star U_0 - \omega_m) & 0 \\ 0 & 0 & \frac{\rho_0 W_0^2}{rc_p} & 0 & -(\omega_m - k^\star U_0) \end{bmatrix}. \tag{50}$$

In the case of homentropic flow, we simply remove the last column and row in $\mathcal{L}^\dagger$, and also note that $\mathcal{L}^\dagger_{3,1} = -\rho_0 \partial/(\partial r)$, since the first two terms cancel. Since $\boldsymbol{\psi}_0$ satisfies the boundary conditions in Eq. (23) we therefore arrive at the adjoint boundary condition:

$$\frac{\rho_0 \Lambda}{\omega Z_H^\ddagger} \mathcal{Y}_1^\star + \mathcal{Y}_3^\star = 0 \text{ at } r = H(X), \tag{51a}$$

$$-\frac{\rho_0 \Lambda}{\omega Z_D^\ddagger} \mathcal{Y}_1^\star + \mathcal{Y}_3^\star = 0 \text{ at } r = D(X). \tag{51b}$$

We now calculate $\boldsymbol{\mathcal{Y}}$ in terms of $\boldsymbol{\psi}_0$, since we have $\mathcal{L}^\dagger \boldsymbol{\mathcal{Y}} = 0$. We find that

$$\mathcal{Y}_1 = \frac{\mathcal{P}_0^\star}{\rho_0 \Lambda^\star}, \tag{52a}$$

$$\mathcal{Y}_2 = \frac{\mathcal{U}_0^\star}{\Lambda^\star} + \mathrm{i} \frac{\partial U_0}{\partial r} \frac{\mathcal{V}_0^\star}{(\Lambda^\star)^2}, \tag{52b}$$

$$\mathcal{Y}_3 = -\frac{\mathcal{V}_0^\star}{\Lambda^\star}, \tag{52c}$$

$$\mathcal{Y}_4 = \frac{\mathcal{W}_0^\star}{\Lambda^\star} + \mathrm{i}\left(\frac{\partial W_0}{\partial r} - \frac{W_0}{r}\right) \frac{\mathcal{V}_0^\star}{(\Lambda^\star)^2}, \tag{52d}$$

$$\mathcal{Y}_5 = \frac{\rho_0 W_0^2}{rc_p} \frac{\mathcal{V}_0^\star}{(\Lambda^\star)^2}, \tag{52e}$$

where $\Lambda^\star = k^\star U_0 - \omega_m$. We then use the fact that

$$\langle \boldsymbol{\mathcal{Y}}, \mathcal{L}\boldsymbol{\psi}_1 \rangle = \langle \mathcal{L}^\dagger \boldsymbol{\mathcal{Y}}, \boldsymbol{\psi}_1 \rangle + \Big[r\rho_0 \mathcal{Y}_1^\star \mathcal{V}_1 + r\mathcal{Y}_3^\star \mathcal{P}_1 \Big]_{H(X)}^{D(X)}, \tag{53}$$

and hence

$$\langle \boldsymbol{\mathcal{Y}}, \boldsymbol{f} \rangle = \Big[r\rho_0 \mathcal{Y}_1^\star \mathcal{V}_1 + r\mathcal{Y}_3^\star \mathcal{P}_1 \Big]_{H(X)}^{D(X)}. \tag{54}$$

B. Applying the boundary condition

Using the boundary condition in Eq. (46), the adjoint boundary conditions in Eq. (51), and the definition of $\mathcal{Y}_1$ in Eq. (52) then gives that

$$\langle \boldsymbol{\mathcal{Y}}, \boldsymbol{f} \rangle = \left[\frac{r\mathcal{P}_0}{\Lambda} \left(\frac{\partial R_j}{\partial X} \mathcal{U}_0 \pm \left[U_0 \frac{\partial}{\partial X} + V_1 \frac{\partial}{\partial r} - \frac{\partial V_1}{\partial r} + \frac{\partial R_j}{\partial X} \frac{\partial U_0}{\partial r} \right] \left(\frac{\mathcal{P}_0}{\mathrm{i}\omega Z_j^\ddagger} \right) \right) \right]_{H(X)}^{D(X)}, \tag{55}$$

This then will give a differential equation in terms of $N(X)$:

$$F_L(X) \frac{\mathrm{d}}{\mathrm{d}X} \left[N^2(X) \right] + G_L(X) N^2(X) = F_R(X) \frac{\mathrm{d}}{\mathrm{d}X} \left[N^2(X) \right] + G_R(X) N^2(X), \tag{56}$$

where F and G are to be determined. Using Eq. (29) we find that

$$G_R = \left[\frac{r\widehat{\mathcal{P}}_0}{\Lambda} \left(\frac{\partial R_j}{\partial X} \widehat{\mathcal{U}}_0 \pm \left[U_0 \frac{\partial}{\partial X} + V_1 \frac{\partial}{\partial r} - \frac{\partial V_1}{\partial r} + \frac{\partial R_j}{\partial X} \frac{\partial U_0}{\partial r} \right] \left(\frac{\widehat{\mathcal{P}}_0}{\mathrm{i}\omega Z_j^\ddagger} \right) \right) \right]_{H(X)}^{D(X)}, \tag{57}$$

$$F_R = \left[\frac{rU_0}{2\Lambda} \frac{\widehat{\mathcal{P}}_0 \widehat{\mathcal{P}}_0}{\mathrm{i}\omega Z_j^\ddagger} \right]_{H(X)}^{D(X)}, \tag{58}$$

$$G_L = \int_{H(X)}^{D(X)} \frac{r\widehat{\mathcal{P}}_0}{\rho_0 \Lambda} \widehat{f}_{\mathcal{P}} + r\left[\frac{\widehat{\mathcal{U}}_0}{\Lambda} - \mathrm{i}\frac{\partial U_0}{\partial r} \frac{\widehat{\mathcal{V}}_0}{\Lambda^2} \right] \widehat{f}_{\mathcal{U}} - \frac{r\widehat{\mathcal{V}}_0}{\Lambda} \widehat{f}_{\mathcal{V}} + r\left[\frac{\widehat{\mathcal{W}}_0}{\Lambda} - \mathrm{i}\left(\frac{\partial W_0}{\partial r} - \frac{W_0}{r} \right) \frac{\widehat{\mathcal{V}}_0}{\Lambda^2} \right] \widehat{f}_{\mathcal{W}} + \frac{\rho_0 W_0^2}{c_p} \frac{\widehat{\mathcal{V}}_0}{\Lambda^2} \widehat{f}_{\mathcal{S}} \mathrm{d}r, \tag{59}$$

and finally

$$F_L = \int_{H(X)}^{D(X)} -\frac{r\widehat{\mathcal{P}}_0}{2\rho_0 \Lambda} \left[\frac{U_0}{c_0^2} \widehat{\mathcal{P}}_0 + \rho_0 \widehat{\mathcal{U}}_0 \right] - \frac{r}{2} \left[\frac{\widehat{\mathcal{U}}_0}{\Lambda} - \mathrm{i}\frac{\partial U_0}{\partial r} \frac{\widehat{\mathcal{V}}_0}{\Lambda^2} \right] \left[\rho_0 U_0 \widehat{\mathcal{U}}_0 + \widehat{\mathcal{P}}_0 \right] + \frac{r\widehat{\mathcal{V}}_0}{2\Lambda} \rho_0 U_0 \widehat{\mathcal{V}}_0$$
$$- \frac{r}{2} \left[\frac{\widehat{\mathcal{W}}_0}{\Lambda} - \mathrm{i}\left(\frac{\partial W_0}{\partial r} - \frac{W_0}{r} \right) \frac{\widehat{\mathcal{V}}_0}{\Lambda^2} \right] \rho_0 U_0 \widehat{\mathcal{W}}_0 - \frac{\rho_0 W_0^2}{2c_p} \frac{\widehat{\mathcal{V}}_0}{\Lambda^2} U_0 \widehat{\mathcal{S}}_0 \mathrm{d}r. \tag{60}$$

Finally, we find that

$$N^2(X) = N_0^2 \exp\left(\int^X \frac{G(\eta)}{F(\eta)} \mathrm{d}\eta \right), \tag{61}$$

where $F = F_L - F_R$, $G = G_R - G_L$ and N_0 is a constant (usually assumed to be one). For homentropic flow, we have $\widehat{f}_{\mathcal{S}} = \widehat{\mathcal{S}}_0 = 0$, and hence the last terms in both F_L and G_L are precisely zero.

C. Hard-walled duct

For a hard-walled duct, there is very little difference in the method, with the only difference being $Z_j = Z_j^\ddagger = \infty$. Thus, F_L and G_L are unchanged, $F_R = 0$ and

$$G_R = \left[\frac{r\widehat{\mathcal{P}}_0}{\Lambda} \frac{\partial R_j}{\partial X} \widehat{\mathcal{U}}_0 \right]_{H(X)}^{D(X)}. \tag{62}$$

VI. Results

A. Initial testing

We consider the same base flow at the inlet as Eq. (17), with the duct still defined in Eq. (18). We still assume $\rho_0(0, r) = 1$. The additional parameters are $m = -2$, $\omega = 10$ and $Z_d = Z_h = 1.5 - 0.5\mathrm{i}$. We firstly solve the leading order eigenvalue problem as X varies, using Chebfun as described in Mathews & Peake.[6] In Figure 2 we plot how the eigenmodes evolve with X, and indicate the initial position when $X = 0$ with filled circles.

In Table 1 we can see how the four most cut-on eigenmodes evolve with X. We can see that two of the eigenmodes become considerably more cut-off as X increases (the first and second rows), while the other two eigenmodes (third and fourth rows) are still quite cut-on as X increases to $X = 2$, with both of these eigenmodes moving left.

Next, we consider the eigenfunctions $\mathcal{P}_0$ for the four most cut-on eigenmodes. To do this required calculating $N(X)$ at each of the eigenmodes under consideration. In the method for calculating $N(X)$ we

	$X = 0$	$X = 0.5$	$X = 1$	$X = 1.5$	$X = 2$
k_1	$5.6413 + 2.3999\mathrm{i}$	$5.4004 + 2.5837\mathrm{i}$	$4.361 + 3.5188\mathrm{i}$	$3.1534 + 4.9112\mathrm{i}$	$2.7948 + 5.3339\mathrm{i}$
k_2	$-10.8203 - 3.1696\mathrm{i}$	$-10.73 - 3.5444\mathrm{i}$	$-10.5046 - 5.5738\mathrm{i}$	$-11.0162 - 9.3414\mathrm{i}$	$-11.5256 - 10.8911\mathrm{i}$
k_3	$8.0404 + 0.6801\mathrm{i}$	$7.9585 + 0.7143\mathrm{i}$	$7.6086 + 0.8493\mathrm{i}$	$7.1572 + 0.9642\mathrm{i}$	$7.0072 + 0.9794\mathrm{i}$
k_4	$-14.9325 - 0.6793\mathrm{i}$	$-15.091 - 0.7271\mathrm{i}$	$-16.1029 - 1.0207\mathrm{i}$	$-18.6113 - 1.563\mathrm{i}$	$-19.9707 - 1.7534\mathrm{i}$

Table 1: Values of the four most cut-on eigenmodes at different values of X.

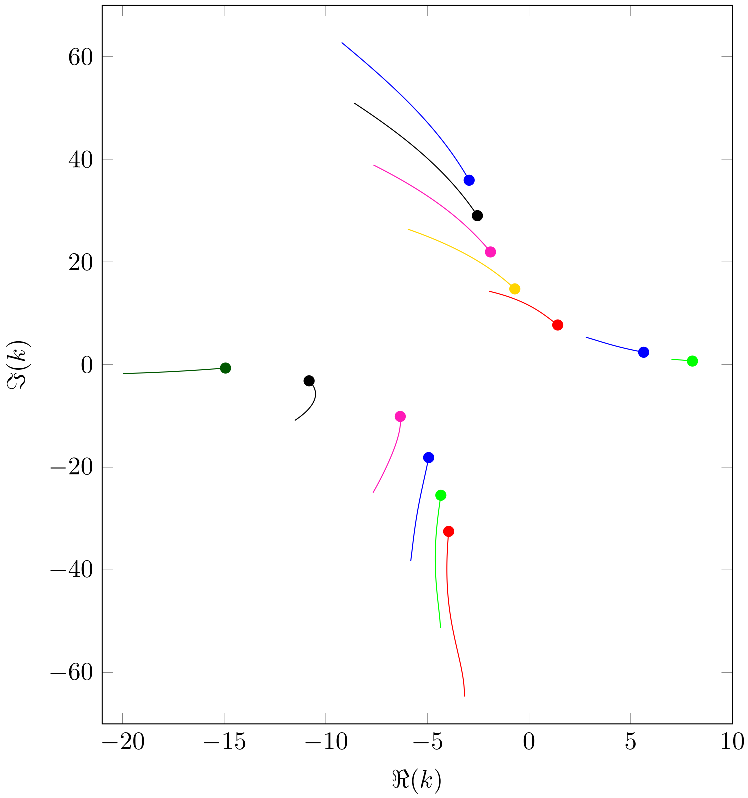

Figure 2: Trajectory of eigenmodes in the complex plane as X varies. The circle indicates the eigenmode at $X = 0$. The base flow is given in Eq. (17), with initial density $\rho_0(r) = 1$ at the inlet, and additionally we have $m = -2$, $\omega = 10$ and $Z_d = Z_h = 1.5 - 0.5\mathrm{i}$.

mostly using Chebfun, although to calculate the X derivatives we used finite differences, and to compute the integration in Eq. (61) we used the trapezium rule. The radial derivatives and radial integration were easily found using Chebfun's automatic differentiation and integration. We had to be careful in choosing the branch cut when calculating the square root in Eq. (61) to avoid crossing it as X varies. In Figure 3 we plot the eigenfunctions $\mathcal{P}_0$ for the eigenmodes k_1 to k_4 in Table 1. In the figure we have normalised the eigenfunction $\widehat{\mathcal{P}}_0(X, r)$ at a particular radial value, before then multiplying by $N(X)$. We can clearly see that the eigenfunction $\mathcal{P}_0$ can differ quite significantly as X varies.

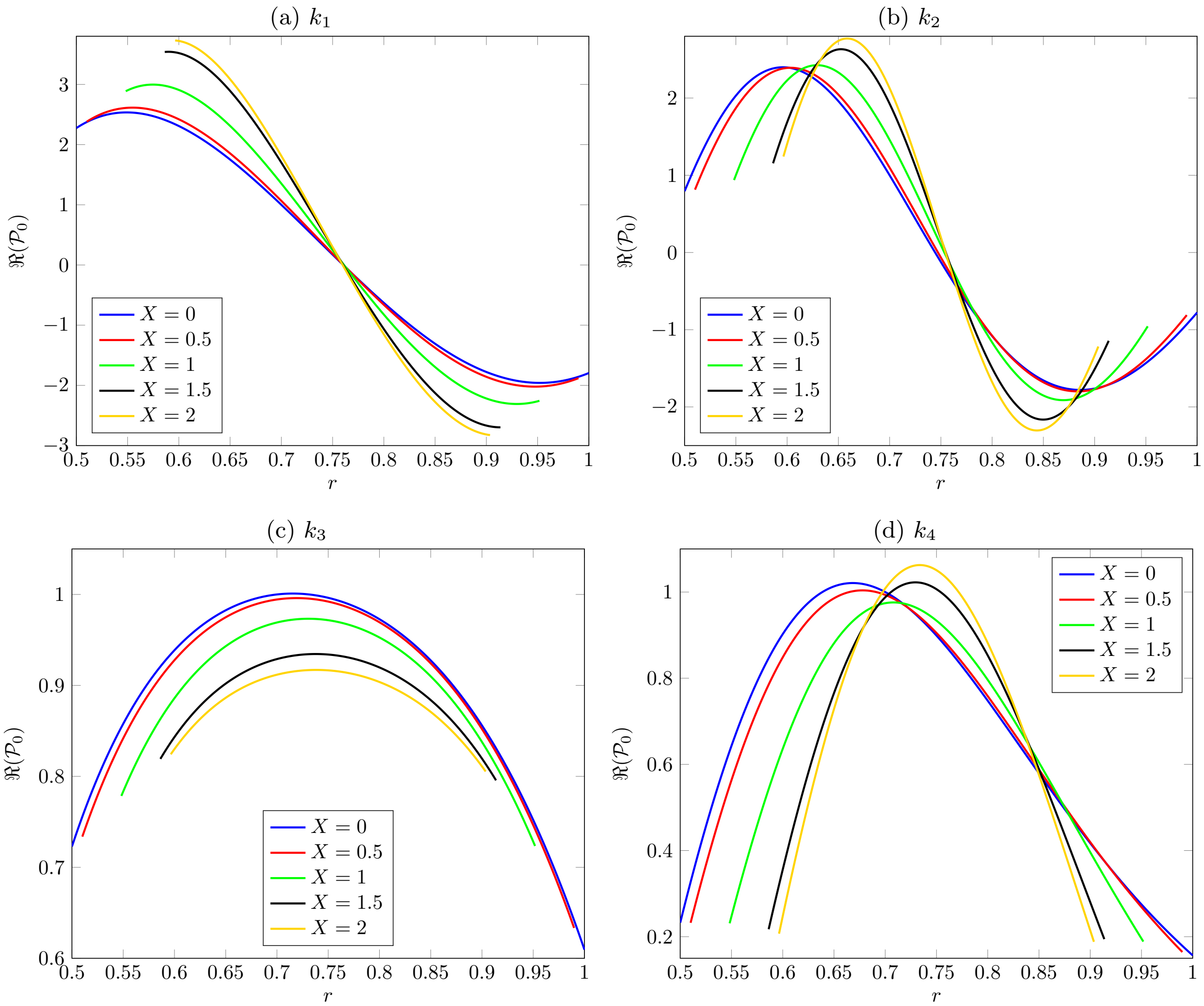

Figure 3: Plot of the real part of the eigenfunctions $\mathcal{P}_0$ for the four most cut-on eigenmodes as X evolves.

B. Realistic swirling flow

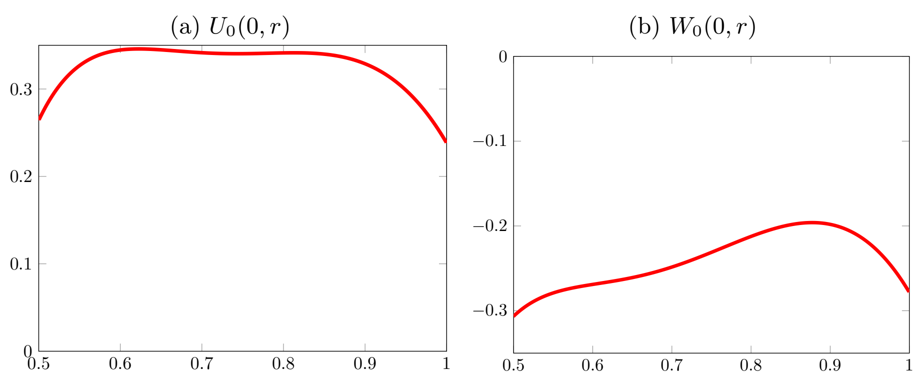

Figure 4: Plot of axial and swirling SDT flow at the inlet.

We now consider realistic swirling flow, in a hard-walled duct. We use the SDT flow from Masson et al,[4] so at the inlet the axial and swirling flows are given by Figure 4. We again prescribe the initial density as $\rho_0(0, r) = 1$, and choose the same duct as the previous case, as given in Eq. (18).

The evolution of the base flow is then given by Figure 5. We see the base flow entropy only varies a small amount. The values of the swirl stay relatively constant throughout the duct, while the axial flow speeds up considerably as X increases.

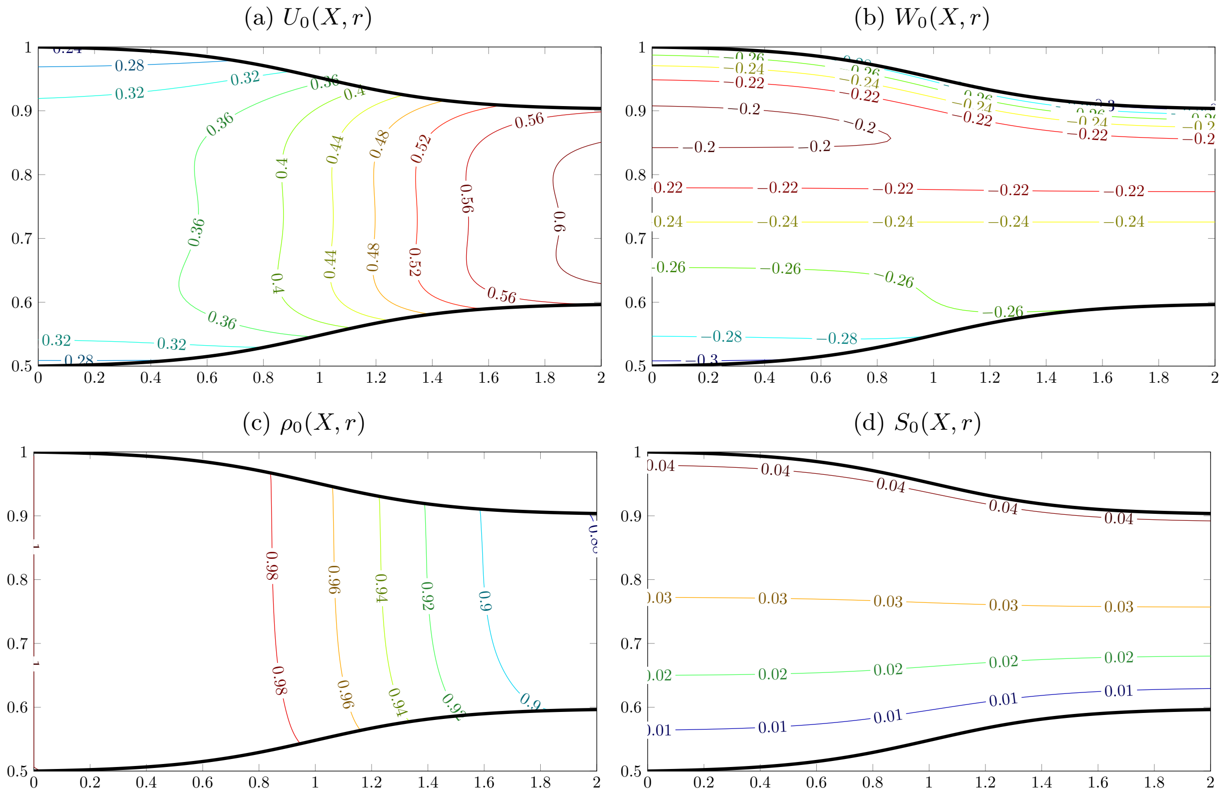

Figure 5: Plot of the SDT base flow with the duct defined in Eq. (18).

We next consider the evolution of the eigenvalues as X varies. The additional parameters that we choose are $m = 3$, $\omega = 10$ and hard walls. The trajectories of the eigenvalues as X varies are given in Figure 6, while Table 2 gives the eigenvalues at particular values of X. We see that the two cut-on modes stay cut-on and

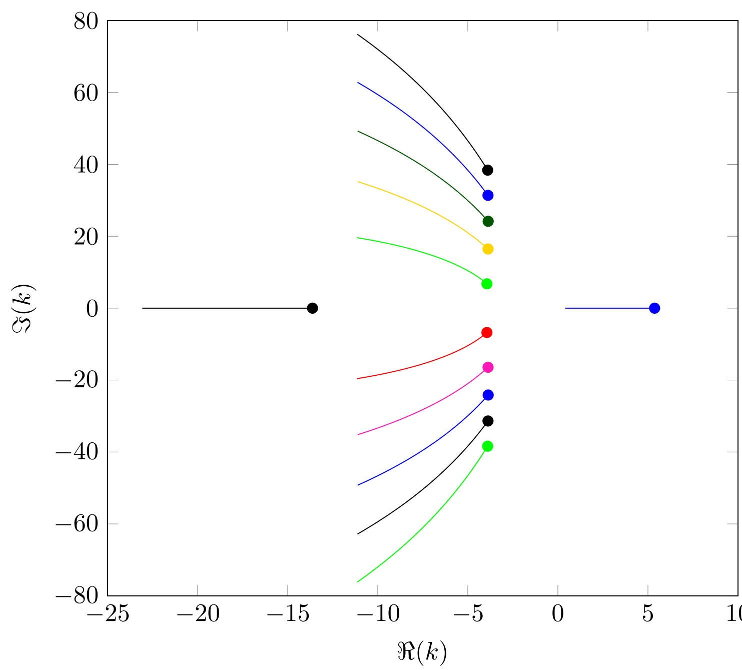

Figure 6: Trajectory of eigenmodes in the complex plane as X varies. The circle indicates the eigenmode at $X = 0$. The base flow is given in Figure 5 and additionally we have $m = 3$, $\omega = 10$ and hard walls.

	$X = 0$	$X = 0.5$	$X = 1$	$X = 1.5$	$X = 2$
k_1	5.3706	5.0939	3.6709	1.2579	0.4071
k_2	$-3.938 \pm 6.7697\mathrm{i}$	$-4.1755 \pm 7.7521\mathrm{i}$	$-5.5316 \pm 11.7862\mathrm{i}$	$-8.9612 \pm 17.4585\mathrm{i}$	$-11.1546 \pm 19.6045\mathrm{i}$
k_3	-13.6199	-13.8201	-15.1064	-19.5271	-23.0702

Table 2: Values of the four most cut-on eigenmodes at different values of X.

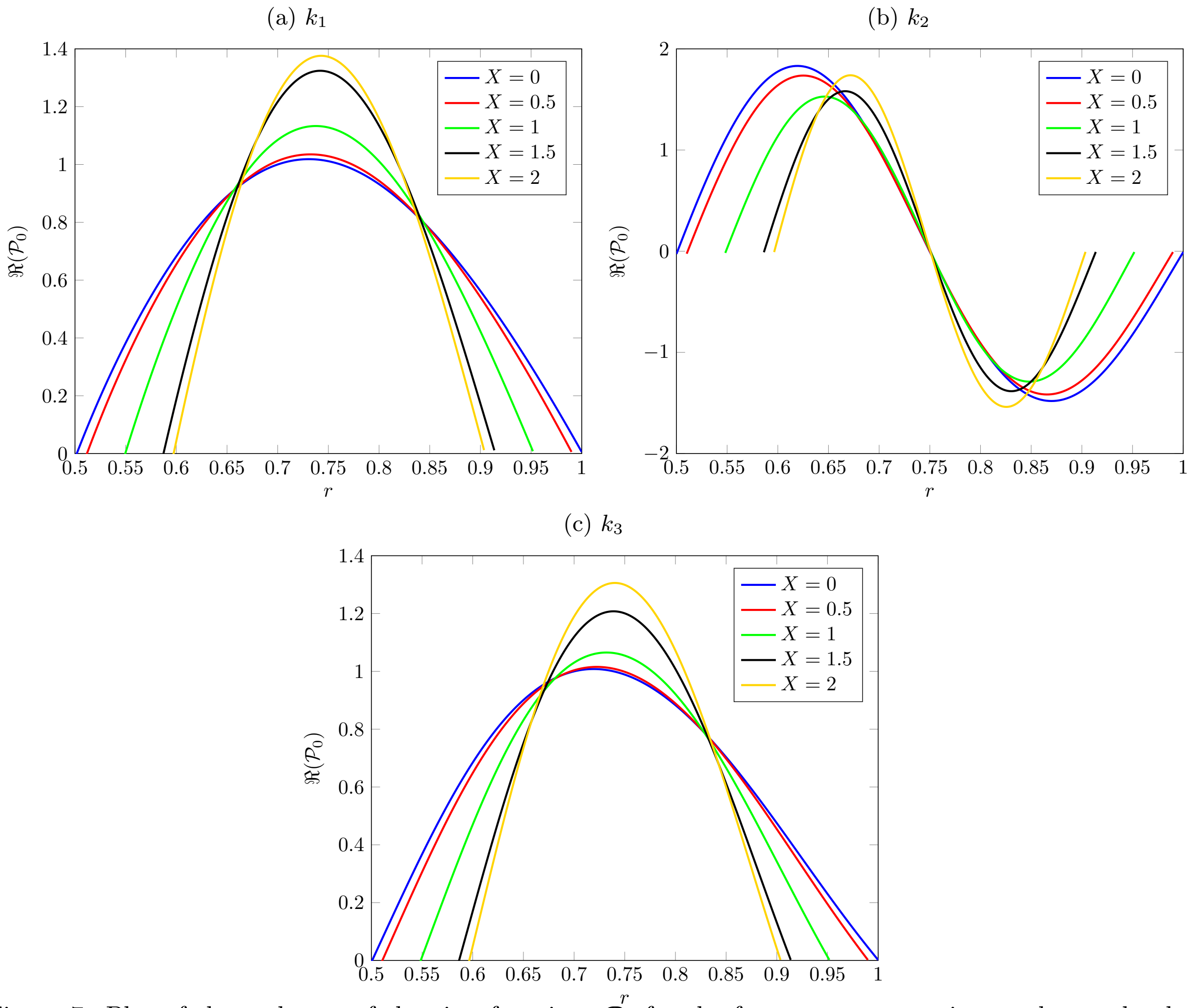

Figure 7: Plot of the real part of the eigenfunctions $\mathcal{P}_0$ for the four most cut-on eigenmodes as the duct evolves.

move to the left, while the cut-off modes all become further cut-off and also move to the left. The movement to the left is due to the axial flow being sped up as it goes though the contracting duct.

Finally, we consider the eigenfunctions $\mathcal{P}_0$, which requires the calculation of the slowly varying function $N(X)$. We calculate this in the same way as previously. In Figure 7 we plot $\mathcal{P}_0$ at five different values of X, in increments of $\Delta X = 0.5$. We again see quite significant variations in the eigenfunctions as X varies, which is to be expected since the eigenmodes moved quite significantly.

VII. Conclusion

In this paper we have re-derived the multiple scales solution to the acoustic disturbances in the case of realistic, swirling isentropic flow. Instead of using the acoustic potential, our solution is explicitly in terms of the pressure and velocity. Our results are valid for any flow and any duct geometry, and we have additionally used the new corrected Myers boundary condition in a curved duct. We have used a hypothesised boundary condition, although we note that to the order we require it (leading order), it is identical to the boundary condition given in Masson et al.[4] The main challenge in deriving the solution was to calculate the base flow and then calculate the slowly varying function. We saw that the slowly varying function varies quite significantly with X and therefore is essential to calculate.

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